ASEN 5014 - Linear Control Design
Homework 5

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Problem 6.32

If $Ax = 0$ has q linearly independent solutions $\mathbf{x}_i$ and $Ax = \mathbf{y}$ has $\mathbf{x}_0$ as a solution, show that:

(a) $\mathbf{x}_c = \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is also a solution of $Ax = 0$,

(b) $\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is a solution of $Ax = \mathbf{y}$.

Instructor's question: What is the dimension of the Right Null Space of A?

Part a)

Answer: We are given that $A\mathbf{x} = 0$ has q linearly independent solutions $\mathbf{x}_i$, for $i = 1, \ldots, q$.

Need to show $A\mathbf{x}_c = 0$, where $\mathbf{x}_c = \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$.

$\mathbf{x}_i$ are solutions to $A\mathbf{x} = 0$, meaning that for each $\mathbf{x}_i$:

$$A\mathbf{x}_i = 0 \quad \forall i = 1, 2, \ldots, q$$

Applying A to $\mathbf{x}_c$:

$$A\mathbf{x}_c = A\left(\sum_{i=1}^{q} \alpha_i \mathbf{x}_i\right)$$

By linearity,

$$A\mathbf{x}_c = \sum_{i=1}^{q} \alpha_i A\mathbf{x}_i$$

We know that $A\mathbf{x}_i = 0$, so

$$A\mathbf{x}_c = \sum_{i=1}^{q} \alpha_i \cdot 0 = 0$$

Therefore, $\mathbf{x}_c$ is a solution to $A\mathbf{x} = 0$.

Part b)

Answer: We are asked to show that $\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is a solution to $A\mathbf{x} = \mathbf{y}$.

We already know that $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{y}$, so

$$A\mathbf{x}_0 = \mathbf{y}$$

From part a, we also know that $A\mathbf{x}_c = 0$, where $\mathbf{x}_c = \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$.

Applying A to $\mathbf{x}_0 + \mathbf{x}_c$ and by linearity:

$$A\mathbf{x} = A\left(\mathbf{x}_0 + \mathbf{x}_c\right) = A\mathbf{x}_0 + A\mathbf{x}_c$$

We know that $A\mathbf{x}_0 = \mathbf{y}$ and $A\mathbf{x}_c = 0$, so

$$A\mathbf{x} = \mathbf{y} + 0 = \mathbf{y}$$

Therefore $\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is a solution to $A\mathbf{x} = \mathbf{y}$.

Instructor's Question

Answer: We are told that $A\mathbf{x} = 0$ has q linearly independent solutions.
By definition, the Right Null Space of M contains all the vectors x such that $Mx = 0$.
i.e $\text{RN}(M) = \{x \in \mathbb{R}^n \mid Mx = 0\}$.
In this case, $M = A$ and q are the solutions such that $A\mathbf{x} = 0$.
Therefore, the $\boxed{\text{dimension of the Right Null Space of } A \text{ is } q.}$

Problem 6.36

Find all nontrivial solutions of $A\mathbf{x} = 0$, i.e., the null space of

$$A = \begin{bmatrix} 26 & 17 & 8 & 39 & 35 \\ 17 & 13 & 9 & 29 & 28 \\ 8 & 9 & 10 & 19 & 21 \\ 39 & 29 & 19 & 65 & 62 \\ 35 & 28 & 21 & 62 & 61 \end{bmatrix}.$$

Answer: We can run the following code in Matlab for easy acccess to the null space of A:

```
% Defining the matrix
A = [26 17 8 39 35;
     17 13 9 29 28;
      8  9 10 19 21;
     39 29 19 65 62;
     35 28 21 62 61];

% Display the null space
disp('The basis for the null space of matrix A is:');
disp(null(A));
```

The output that we obtain is:

```
The basis for the null space of matrix A is:
-0.6211    0.2577    0.0225
 0.4899    0.8165    0.1220
-0.4884    0.2774   -0.4336
 0.3619   -0.3400   -0.5846
-0.0682   -0.2726    0.6745
```

This means we have 3 columns in the basis of the null space of A. The null space of A is spanned by the orthonormal (`null` in MATLAB returns orthonormal basis) vectors:

$$\mathbf{x} = \left\{ \begin{bmatrix} -0.6211 \\ 0.4899 \\ -0.4884 \\ 0.3619 \\ -0.0682 \end{bmatrix}, \begin{bmatrix} 0.2577 \\ 0.8165 \\ 0.2774 \\ -0.3400 \\ -0.2726 \end{bmatrix}, \begin{bmatrix} 0.0225 \\ 0.1220 \\ -0.4336 \\ -0.5846 \\ 0.6745 \end{bmatrix} \right\}$$

These are the nontrivial solutions of $A\mathbf{x} = 0$.

Problem 6.38

Given that $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$. Measurements give $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find the least-squares estimate for x. Use a sketch in the y_1, y_2 plane to indicate the geometrical interpretation.

Answer: We are given the equation: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$.
We are also given the measurements: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

We can begin by subsutiuting the given values into the equation:

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Thsi gives us a system of equations:

$$3 = 2x_1 + e_1$$
$$4 = 2x_2 + e_2$$

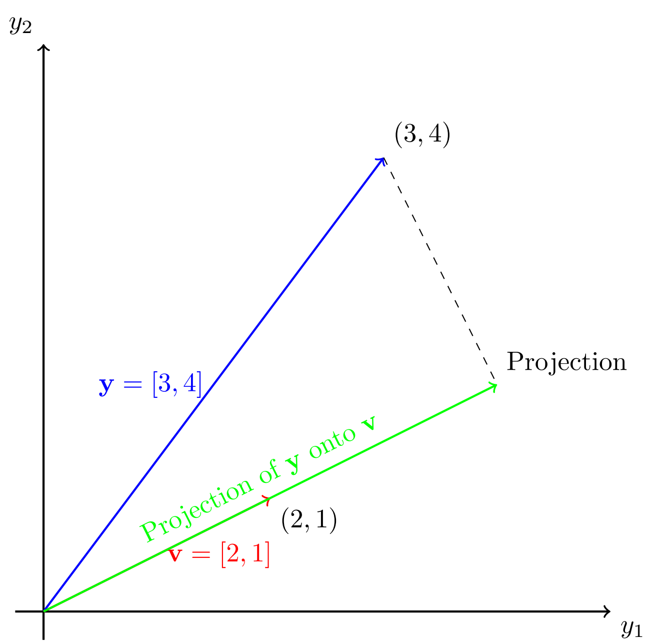
Our goal with Least Squares is to minimize the error vector e. The easiest way to do this is to project the vector $y = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ on to the column space of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
The formula for the least squares of x in this situation is given by:

$$x = \frac{v^\top y}{v^\top v}$$

Calculating the dot products:

$$v^\top y = 2(3) + 1(4) = 10$$
$$v^\top v = 2^2 + 1^2 = 5$$

Thus the least squares can be calculated to be $x = \frac{10}{5} = 2$.



Problem 6.41

Problem 6.40 - A physical device is shown in Figure 6.10. It is believed that the output y is linearly related to the input u. That is $y = au + b$. What are the values of a and b if the following data are taken?

u	y
2	5
-2	1

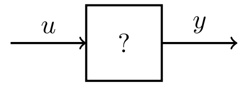


Figure 1: Figure 6.10

The same device as in Problem 6.40 is considered. One more set of readings is taken as

$$u = 5, \quad y = 7$$

Find a least-squares estimate of a and b. Also, find the minimum mean-squared error in this straight line fit to the three points.

Answer: We are given a linear equation $y = au + b$, and the following data points:

$$(2, 5), \quad (-2, 1), \quad (5, 7)$$

We can write the following equations from the data points:

$$\begin{aligned} 5 &= 2a + b \\ 1 &= -2a + b \\ 7 &= 5a + b \end{aligned}$$

We can the set up a Matrix form of the above equations:

$$\begin{bmatrix} 2 & 1 \\ -2 & 1 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$$

This equation is basically of the form $\mathbf{U} \cdot \vec{w} = \mathbf{Y}$.

The goal of Least Squares estimate is to find the values of $\vec{w}$ that minimizes the error. In toher words, we want to find the values of a and b that minimises the error between actual values of y and the predicted values of $\hat{y}$.
The error is given by:

$$\text{Error} = \mathbf{Y} - \mathbf{U} \cdot \vec{w}$$

We want the error to be minimum, so we want to minimize the square of the error. The square of the error is given by:

$$\text{Error}^2 = (Y - U \cdot \mathbf{w})^\top \cdot (Y - U \cdot \mathbf{w})$$
$$\text{Error}^2 = Y^\top Y - Y^\top U\mathbf{w} - (U\mathbf{w}^\top)Y + (U\mathbf{w}^\top)(U\mathbf{w})$$
$$\text{Error}^2 = Y^\top Y - 2Y^\top U\mathbf{w} + \mathbf{w}^\top U^\top U\mathbf{w}$$

We can the differentiate the above equation with respect to $\mathbf{w}$ and set it to zero to find the minimum value of $\mathbf{w}$.

$$\frac{\mathrm{d}\left(\text{Error}^2\right)}{\mathrm{d}\mathbf{w}} = -2U^\top Y + 2U^\top U\mathbf{w} = 0$$

Dividing both sides by 2, we get:

$$U^\top Y = U^\top U\mathbf{w}$$

We can solve the above equation to find the values of $\mathbf{w}$.

$$\mathbf{w} = (U^\top U)^{-1} U^\top Y$$

The following Matlab code can be used to find the values of a and b:

```
% Data points from question 6.41
% Matrix of u values and adding a columnof 1s for the b in the equations
U = [2 1; -2 1; 5 1];
% Corresponding y-values
Y = [5; 1; 7];

% Least-squares estimate of [a; b]
% NOTE: A\B in MATLAB is the equivalent of inv(A) * B
w_star = (U' * U) \ (U' * Y);

% Display the results
a = w_star(1);
b = w_star(2);
fprintf('Least-squares estimate for a: %f\n', a);
fprintf('Least-squares estimate for b: %f\n', b);

% Predicted values
Y_hat = U * w_star;

% Mean-squared error
MSE = mean((Y - Y_hat).^2);

% Display the result
fprintf('Mean-squared error: %f\n', MSE);
```

The output we obtain is:

```
Least-squares estimate for a: 0.864865
Least-squares estimate for b: 2.891892
Mean-squared error: 0.072072
```

One thing to note in the script is that we also calculated the mean squared error. The mean squared error is given by:

$$\text{Mean Squared Error} = \frac{1}{N}\sum_{i=1}^{N}(y_i - \hat{y}_i)^2$$

where $\hat{y} = au + b$ or $\hat{y} = \mathbf{U} \cdot \mathbf{w}$.

To answer the question, the least-squares estimate of a and b are 0.864865 and 2.891892 respectively. The minimum mean-squared error in this straight line fit to the three points is 0.072072.

Problem 6.45

An empirical theory used by many distance runners states that the time T_i required to race a distance D_i can be expressed as $T_i = C(D_i)^\alpha$, where C and α are constants for a given person, determined by lung capacity, body build, etc. Obtain a least-squares fit to the following data for one middle-aged jogger. (Convert to a linear equation in the unknowns C and α by taking the logarithm of the above expression.) Predict the time for one mile.

Time	Distance
185 min	26.2 mi
79.6 min	12.4 mi
60 min	9.5 mi
37.9 min	6.2 mi
11.5 min	2 mi

Answer: We are given $T_i = C(D_i)^\alpha$.

In order to get a linear equation from the above expression, we take the logarithm of both sides.

$$\log(T_i) = \log(CD_i^\alpha) = \log C + \log(D_i^\alpha) = \log C + \alpha \log(D_i)$$

$$\log(T_i) = \log C + \alpha \log(D_i)$$

This equation is of the form $y = mx + b$, where $y = \log(T_i)$, $x = \log(D_i)$, $m = \alpha$, and $b = \log C$.

For our purposes, we can then take logartithms of the given data and then solve the least squares formula:

$$\mathbf{w} = (X^\top X)^{-1} X^\top Y$$

where $X = \begin{bmatrix} \log(D_1) & 1 \\ \log(D_2) & 1 \\ \vdots & \vdots \\ \log(D_n) & 1 \end{bmatrix}$, $Y = \begin{bmatrix} \log(T_1) \\ \log(T_2) \\ \vdots \\ \log(T_n) \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} \alpha \\ \log C \end{bmatrix}$.

We can then finally predict the time for one mile by using the formula $T_i = C(D_i)^\alpha$.

$$T_{\text{mile}} = C(1)^\alpha = C$$

This can all be done with the following MATLAB code:

```matlab
% Making vectors of the data, T(min) and D(miles)
T = [185, 79.6, 60, 37.9, 11.5];
D = [26.2, 12.4, 9.5, 6.2, 2];

% Take logarithms of both time and distance to linearize
y_i = log(T);
x_i = log(D);

% Setting up the least-squares problem: log(T) = log(C) + alpha * log(D)
% X is a matrix with the log(D) and a column of ones to account for the intercept log(C)
X = [x_i', ones(length(x_i), 1)];

% Solving the least squares problem: w = inv(X' * X) * X' * y
```

```
% Note: we transpose the y vector to make it a column vector
w = (X' * X) \ (X' * y_i');

% The slope (alpha) and the intercept (log(C)) are the coefficients of the
    linear model that are obtained from the least-squares solution w
alpha = w(1);
logC = w(2);

% TO go back from a linear model to the original model, we need to take
   the exponential
C = exp(logC);

% Display results
fprintf('Least-squares estimate for alpha: %f\n', alpha);
fprintf('Least-squares estimate for C: %f\n', C);

% Predict the time for one mile, see equation in document
% Since D = 1 mile, we have T = C * (1^alpha)
T_one_mile = C * (1^alpha);

% Display the predicted time for one mile
fprintf('Predicted time for one mile: %f minutes\n', T_one_mile);
```

We are given the following output:

```
Least-squares estimate for alpha: 1.076946
Least-squares estimate for C: 5.370498
Predicted time for one mile: 5.370498 minutes
```

Therefore, the least-squares estimate for $\boxed{\alpha \text{ is } 1.076946}$, the least-squares estimate for $\boxed{C \text{ is } 5.370498}$, and the predicted $\boxed{\text{time for one mile is 5.370498 minutes.}}$